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Extra Capitulation and Central Extensions

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Let L be a cyclic unramified extension of the number field K, with $G := \mathrm{Gal}(L/K)$, and $L^{(1)}$ the Hilbert class field of L. The central object of studying those ideals of K which become principal, i.e., *capitulate*, has been $H^1(G, E_L)$, where E_L denotes the group of global units of L. However, if one lets C_L and U_L denote the idele class group of L and the group of unit ideles, respectively, there is an isomorphism $H^{i+1}(G, E_L) = H^i(G, U_L/E_L)$, and U_L/E_L has the advantage of being isomorphic to an idele class subgroup of C_L; this is our basic tool. In this paper, we study "extra" capitulation, that is, whenever there is more capitulation than one would normally expect. More precisely, we show that there is a nontrivial *ramified* central extension of $L^{(1)}M/K$, with M some abelian extension of K, exactly when there is extra capitulation. © 1995 Academic Press, Inc.

INTRODUCTION

An ideal $\mathfrak{p}$ of a number field is said to *capitulate* in the abelian extension L of K, if $\mathfrak{p}$ is not principal in K, but is principal in L. Hilbert's Theorem 94 says that if L is cyclic and unramified over K, then there are at least $[L:K]$ ideals which capitulate in L. There are many cases, however, where a larger group of ideals capitulates in L. We will call this *extra capitulation*. Let $G := \mathrm{Gal}(L/K)$, and C_L denote the ideal class group of L. When G is cyclic, $G = \langle g \rangle$, it follows from the exact sequence

$$1 \to (C_L)^G \to C_L \xrightarrow{\cdot(1-g)} C_L \to \frac{C_L}{(C_L)^{1-g}} \to 1$$

that $|C_L/(C_L)^{1-g})| = |(C_L)^G|$. It follows from the genus theory of number fields that $|(C_L)^G| = |C_L/((C_L)^{1-g})| = |C_K|/|G|$. Hence, extra capitulation occurs exactly when $j\colon C_K \to (C_L)^G$ is not surjective. That is, for each "extra" ideal which capitulates in L, there must exist an "extra" ambiguous

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ideal class. Let $L^{(1)}$ denote the Hilbert class field of L. In this paper we show that $(C_L)^G/j(C_K)$ is isomorphic to the subgroup of the Galois group of a central extension of $L^{(1)}/K$ which is generated by the inertia groups of the extension. That is, in order to measure the "extra" capitulation, it helps to look at *ramified* extensions! We close by giving a related group extension result.

1. The Theorems

Let L be a cyclic unramified extension of the number field K, of degree n, and let $G = \mathrm{Gal}(L/K)$. Let Cl_L denote the idele class group of L, U_L the unit ideles of L, C_L the ideal class group of L, and E_L the global units of L. An arbitrary idele will be denoted by (x), while its idele class in Cl_L will be labeled $\overline{(x)}$. $L^{(1)}$ will denote the Hilbert field of L. Θ will denote the reciprocity map $\Theta\colon \mathrm{Cl}_L \to \mathrm{Gal}(L^{\mathrm{ab}}/L)$, where L^{ab} denotes the maximal abelian extension of L.

Lemma 1. *Extra capitulation occurs if and only if* $H^1(G, U_L/E_L) \neq (1)$.

Proof. It is well known that the subgroup of C_K which capitulates in L is isomorphic to $H^1(G, E_L)$, where E_L is the group of global units of L, and that when L is unramified over K, the Herbrandt quotient $h(E_L) = |H^1(G, E_L)|/|H^2(G, E_L)|$ of E_L is n. From this we see that the capitulation is "good," that is, $|H^1(G, E_L)| = n$, if and only if $H^2(G, E_L) = (1)$. From the exact sequence $1 \to E_L \to U_L \to U_L/E_L \to 1$, and the fact that $\hat{H}^i(G, U_L) = (1)$ because L is unramified over K, we have $H^1(G, U_L/E_L) \cong H^2(G, E_L)$. So we can conclude that there is extra capitulation if and only if $H^1(G, U_L/E_L) \neq (1)$. ∎

Lemma 2. $(\mathrm{Cl}_L)^{1-g}/(U_L/E_L)^{1-g}$ *is isomorphic to* $\mathrm{Gal}(L'/K^{\mathrm{ab}})$, *where L' is the maximal central extension of $L^{(1)}K^{\mathrm{ab}}/K$ in L^{ab}.*

Proof. Let $S = \mathrm{Gal}(L^{\mathrm{ab}}/K)$ and $H = \mathrm{Gal}(L^{\mathrm{ab}}/L^{(1)})$. Then as K^{ab} is the fixed field of $[S, S]$ and L' is the fixed field of $[H, S]$, we have $\mathrm{Gal}(L'/K^{\mathrm{ab}})$ isomorphic to $[S, S]/[H, S]$, where $[S, S]$ is being identified with $\mathrm{Gal}(L^{\mathrm{ab}}/SK^{\mathrm{ab}})$ and $[H, S]$ with $\mathrm{Gal}(L^{\mathrm{ab}}/L')$. As both

$$\Theta\colon (\mathrm{Cl}_L)^{1-g} \to [S, S] \cong \mathrm{Gal}(L^{\mathrm{ab}}/K^{\mathrm{ab}})$$

and

$$\Theta\colon \left(\frac{U_L}{E_L}\right)^{1-g} \to [H, S] \cong \mathrm{Gal}(L^{\mathrm{ab}}/L')$$

are surjective, and as any element in the kernel of either of these maps would have to have trivial image in $\mathrm{Gal}(L^{ab}/L)$, we have

$$\mathrm{Gal}(L'/K^{ab}) \cong \frac{(\mathrm{Cl}_L)^{1-g}/((\mathrm{Cl}_L)^{1-g} \cap D(L))}{(U_L/E_L)^{1-g}/(U_L/E_L)^{1-g} \cap D(L)},$$

where $D(L)$ is the kernel of Θ; that is, $D(L)$ is the connected component of the identity in Cl_L. Let N denote the usual norm map $N\colon \mathrm{Cl}_L \to \mathrm{Cl}_K$. It is clear that if any idele class $\overline{(y)}$ is contained in $(\mathrm{Cl}_L)^{1-g} \cap D(L)$, then it has trivial norm under the norm map N. Let N_d denote the restriction of N to $D(L)$. Then we have $N_d(\overline{(y)}) = 1$ in Cl_K. Note that the image of $\overline{(y)}$ is actually contained in $D(K)$. It is known that $\hat{H}^{-1}(G, D_L) = (1)$ (see [1, p. 33]). This means that $\mathrm{Ker}\, N_d = \{\overline{(y)} \mid \overline{(y)} = \overline{(x)}^{1-g}, \text{ with } (x) \in D(L)\}$. That is, all $\overline{(y)} \in D(L)$ of trivial norm under the map $N_d\colon D(L) \to D(K)$ are of the form $\overline{(x)}^{1-g}$, where $\overline{(x)} \in D(L)$. So, we conclude that

$$((\mathrm{Cl}_L)^{1-g} \cap D(L)) \subset (D(L))^{1-g}.$$

We will show next that $(D(L))^{1-g} \subset (U_L/E_L)^{1-g}$. In fact, we claim that if $\overline{(x)} \in D(L)$, it must be possible to represent the class of $\overline{(x)}$ in Cl_L with a unit idele (u). We do this by contradiction. Assume that is not possible. Then $\overline{(x)}$ would have nontrivial image in $\mathrm{Cl}_L/\overline{U_L} \cong (U_L/E_L) \cong C_L =$ ideal class group of L. Since the *ideal* represented by $\overline{(x)}$ would have nontrivial image in $\mathrm{Gal}(L^{(1)}/L)$ via the isomorphism $C_L \to \mathrm{Gal}(L^{(1)}/L)$. $e(\overline{(x)})$ would have nontrivial image in $\mathrm{Gal}(L^{(1)}/L)$, contradicting $(x) \in D(L)$. So, our initial assumption must have been false, and in fact $\overline{(x)} = \overline{(u)}$, for some unit idele (u). It is then clear that $D(L)^{1-g} \subset (U_L/E_L)$.

Recall that $\overline{(y)}$ represents an arbitrary element in $(\mathrm{Cl}_L)^{1-g} \cap D(L)$. From the results of the last two paragraphs, we see that $\overline{(y)} \in (U_L/E_L)^{1-g}$ (that is, in the image of $(U_L/E_L)^{1-g} \to \mathrm{Cl}_L$), implying that

$$(\mathrm{Cl}_L)^{1-g} \cap D(L) \subset \left(\frac{U_L}{E_L}\right)^{1-g},$$

further implying that

$$\frac{(\mathrm{Cl}_L)^{1-g}/((\mathrm{Cl}_L)^{1-g} \cap D(L))}{(U_L/E_L)^{1-g}/((U_L/E_L)^{1-g} \cap D(L))} \cong \frac{(\mathrm{Cl}_L)^{1-g}}{(U_L/E_L)^{1-g}}. \quad \blacksquare$$

THEOREM 3. *$H^1(G, U_L/E_L)$ is isomorphic to $\mathrm{Gal}(L'/K^{ab}L^{(1)})$, where L' denotes the maximal central extension of $L^{(1)}K^{ab}/K$ contained in L^{ab}.*

Proof. Since L/K is cyclic, $H^1(G, U_L/E_L) \cong \hat{H}^{-1}(G, U_L/E_L)$, where $\hat{H}^{-1}(G, U_L/E_L)$ denotes the (-1)st Tate group. By definition, $\hat{H}^{-1}(G, U_L E_L)$

is defined to be $(\operatorname{Ker} N_u)/(U_L/E_L)^{1-g}$, where N_u is the norm map on unit ideles associated with L/K, and g is any generator of G. However, N_u is the restriction of the usual norm map N on Cl_L. Since G is cyclic, $\hat{H}^{-1}(G, \mathrm{Cl}_L) = (\operatorname{Ker} N)/(\mathrm{Cl}_L)^{1-g} \cong H^1(G, \mathrm{Cl}_L) = (1)$. So, every idele class (y) whose norm is trivial is of the form $\overline{(y)} = \overline{(x)}^{1-g}$ (in Cl_L) for some idele class $\overline{(x)}$. Since unit idele classes are idele classes, any $\overline{(u)} \in \operatorname{Ker} N_u$ is of the form $\overline{(u)} = \overline{(x)}^{1-g}$ (in Cl_L) so that we may conclude that $\operatorname{Ker} N_u = (\mathrm{Cl}_L)^{1-g} \cap U_L/E_L$, where we identify U_L/E_L with its isomorphic image in Cl_L.

Identifying $(\mathrm{Cl}_L)^{1-g}/(U_L/E_L)^{1-g}$ with $\mathrm{Gal}(L'/K^{\mathrm{ab}})$, we need to prove that the fixed field of the image of $(\mathrm{Cl}_L)^{1-g} \cap U_L/E_L$ in $(\mathrm{Cl}_L)^{1-g}/(U_L/E_L)^{1-g}$ is $L^{(1)}K^{\mathrm{ab}}$. However, any element of $\Theta(U_L/E_L)$ fixes $L^{(1)}$, so that any element of $\Theta((\mathrm{Cl}_L)^{1-g} \cap (U_L/E_L))$ fixes $L^{(1)}K^{\mathrm{ab}}$. That is (with a slight abuse of notation),

$$\frac{(\mathrm{Cl}_L)^{1-g} \cap (U_L/E_L)}{(U_L/E_L)^{1-g}} \subset \mathrm{Gal}(L'/L^{(1)}K^{\mathrm{ab}}).$$

Also,

$$\frac{(\mathrm{Cl}_L)^{1-g}}{(\mathrm{Cl}_L)^{1-g} \cap (U_L/E_L)} \cong \frac{(\mathrm{Cl}_L)^{1-g}\,(U_L/E_L)}{(U_L/E_L)} \cong \mathrm{Gal}(L^{(1)}/K^{(1)}),$$

because the fixed field of $\Theta(U_L/E_L)$ is $L^{(1)}$ and the fixed field of $\Theta(\mathrm{Cl}_L)^{1-g}$ is K^{ab}, and both the top and bottom of the fraction contain the kernel of Θ. In addition, it is clear that $|\mathrm{Gal}(L^{(1)}/K^{(1)})| = |\mathrm{Gal}(L^{(1)}K^{\mathrm{ab}}/K^{\mathrm{ab}})|$. We then have

$$\begin{aligned}
|\mathrm{Gal}(L'/K^{\mathrm{ab}})| &= \left|\frac{(\mathrm{Cl}_L)^{1-g}}{(U_L/E_L)^{1-g}}\right| \\
&= \left|\frac{(\mathrm{Cl}_L)^{1-g}}{(\mathrm{Cl}_L)^{1-g} \cap (U_L/E_L)}\right| \cdot \left|\frac{(\mathrm{Cl}_L)^{1-g} \cap (U_L/E_L)}{(U_L/E_L)^{1-g}}\right| \\
&= |\mathrm{Gal}(L^{(1)}K^{\mathrm{ab}}/K^{\mathrm{ab}})| \cdot \left|\frac{(\mathrm{Cl}_L)^{1-g} \cap (U_L/E_L)}{(U_L/E_L)^{1-g}}\right|.
\end{aligned}$$

Since we also have

$$|\mathrm{Gal}(L'/K^{\mathrm{ab}})| = |\mathrm{Gal}(L'/L^{(1)}K^{\mathrm{ab}})| \cdot |\mathrm{Gal}(L^{(1)}K^{\mathrm{ab}}/K^{\mathrm{ab}})|$$

it then follows that

$$\left|\frac{(\mathrm{Cl}_L)^{1-g} \cap (U_L/E_L)}{(U_L/E_L)^{1-g}}\right| = |\mathrm{Gal}(L'/L^{(1)}K^{\mathrm{ab}})|.$$

Since we saw above that $((\mathrm{Cl}_L)^{1-g} \cap (U_L/E_L))/(U_L/E_L)^{1-g} \subset \mathrm{Gal}(L'/L^{(1)}K^{\mathrm{ab}})$, and we see here that they have the same order, they must in fact be equal. That is,

$$H^1\left(G, \frac{U_L}{E_L}\right) \cong \frac{(\mathrm{Cl}_L)^{1-g} \cap (U_L/E_L)}{(U_L/E_L)^{1-g}} \cong \mathrm{Gal}(L'/L^{(1)}K^{\mathrm{ab}}). \quad \blacksquare$$

LEMMA 4. *There is a realization of $H^1(G, U_L/E_L)$ as a Galois group of fields F_1/F_2, where each F_i is a finite abelian extension of L.*

Proof. This is standard profinite group theory (see [1, p. 31]). We include the proof for some measure of completeness. Since $|\mathrm{Gal}(L'/K^{\mathrm{ab}})| = [L' : K^{\mathrm{ab}}]$, the group $\mathrm{Gal}(L^{\mathrm{ab}}/L')$ is closed in $\mathrm{Gal}(L^{\mathrm{ab}}/K^{\mathrm{ab}})$. Since this index is finite, it is also open in it. So, there exists an open normal subgroup T of $\mathrm{Gal}(L^{\mathrm{ab}}/K)$ such that $T \cap \mathrm{Gal}(L^{\mathrm{ab}}/K^{\mathrm{ab}}) \subset \mathrm{Gal}(L^{\mathrm{ab}}/L')$. This implies that $T \cap \mathrm{Gal}(L^{\mathrm{ab}}/L^{(1)}K^{\mathrm{ab}}) \subset \mathrm{Gal}(L^{\mathrm{ab}}/L')$ as well. This then implies that the fixed fields F_1, F_2, and F_3 of $T\,\mathrm{Gal}(L^{\mathrm{ab}}/L')$, $T\,\mathrm{Gal}(L^{\mathrm{ab}}/L^{(1)}K^{\mathrm{ab}})$, and $T\,\mathrm{Gal}(L^{\mathrm{ab}}/K^{\mathrm{ab}})$ are all finite extensions of L (and contained in L^{ab}, hence abelian over L), and that

$$\begin{aligned}
\mathrm{Gal}(F_1/F_2) &\cong \frac{T\,\mathrm{Gal}(L^{\mathrm{ab}}/L^{(1)}K^{\mathrm{ab}})}{T\,\mathrm{Gal}(L^{\mathrm{ab}}/L')} \\
&\cong \frac{\mathrm{Gal}(L^{\mathrm{ab}}/L^{(1)}K^{\mathrm{ab}})}{(T \cap \mathrm{Gal}(L^{\mathrm{ab}}/L^{(1)}K^{\mathrm{ab}})) \cdot \mathrm{Gal}(L^{\mathrm{ab}}/L')} \\
&= \frac{\mathrm{Gal}(L^{\mathrm{ab}}/L^{(1)}K^{\mathrm{ab}})}{\mathrm{Gal}(L^{\mathrm{ab}})/L')}, \\
&\qquad \text{since} \quad T \cap \mathrm{Gal}(L^{\mathrm{ab}}/L^{(1)}K^{\mathrm{ab}}) \subset \mathrm{Gal}(L^{\mathrm{ab}}/L'), \\
&\cong \mathrm{Gal}(L'/L^{(1)}K^{\mathrm{ab}}) \cong H^1\left(G, \frac{U_L}{E_L}\right), \qquad \text{by Theorem 3.}
\end{aligned}$$

Similarly,

$$\mathrm{Gal}(F_1/F_3) \cong \frac{(\mathrm{Cl}_L)^{1-g}}{(U_L/E_L)^{1-g}}. \quad \blacksquare$$

THEOREM 5. *In any finite realization of $H^1(G, U_L/E_L)$ as described in Lemma 4, $H^1(G, U_L/E_L)$ is realized as the subgroup of* $\mathrm{Gal}(F_1/F_3)$ *which is generated by the inertia groups of* $\mathrm{Gal}(F_1/L)$ *contained in* $\mathrm{Gal}(F_1/F_3)$.

Proof. From Lemma 4, we say that there was a surjection Θ: $(\mathrm{Cl}_L)^{1-g} \to \mathrm{Gal}(F_1/F_3)$. There is always a surjection from U_L/E_L onto the subgroup generated by the inertia groups of those primes which ramify in a given abelian extension of L. Our description of $H^1(G, U_L/E_L)$ as $((\mathrm{Cl}_L)^{1-g} \cap (U_L/E_L))/(U_L/E_L)^{1-g}$ along with the surjection then shows that $H^1(G, U_L/E_L)$ maps isomorphically into the subgroup of $\mathrm{Gal}(F_1/F_3)$ which is generated by the inertia groups contained in $\mathrm{Gal}(F_1/F_3)$. The map is necessarily surjective since F_2/F_3 is unramified everywhere. ∎

Since G is cyclic, the problem of finding such central extensions may be solved by finding different ways of "lifting" the generator g to an element of $\mathrm{Gal}(M/K)$, where M is some ramified extension of $L^{(1)}$, that is, finding an element $g_M \in \mathrm{Gal}(M/K)$ such that $g_M|_L = g$. The following theorem seems to support that view:

THEOREM 6. *There is extra capitulation in L/K if and only if there is at least one extension of C_L by G distinct from the classes of extensions generated by the class given by* $\mathrm{Gal}(L^{(1)}/K)$.

Proof. From the exact sequence $1 \to U_L/E_L \to \mathrm{Cl}_L \to C_L \to 1$ we get the exact cohomology sequence

$$\hat{H}^0(G, U_L/E_L) \to \hat{H}^0(G, \mathrm{Cl}_L) \to \hat{H}^0(G, C_L) \to H^1(G, U_L/E_L) \to (1).$$

As we have seen, there is extra capitulation if and only $H^1(G, U_L/E_L) \neq (1)$, which the above sequence shows is true if and only if

$$\hat{H}^0(G, \mathrm{Cl}_L) \xrightarrow{f} \hat{H}^0(G, C_L)$$

is not surjective. However, as G is cyclic, $\hat{H}^0(G, \mathrm{Cl}_L) \cong H^2(G, \mathrm{Cl}_L)$, and $\hat{H}^0(G, C_L) \cong H^2(G, C_L)$. However, $H^2(G, \mathrm{Cl}_L)$ is always a cyclic group, generated by the fundamental class $u_{L/K}$, and by [2, Theorem 11.5(ii)], the image of $u_{L/K}$ in $H^2(G, C_L)$ is the class of the group extension given by $\mathrm{Gal}(L^{(1)}/K)$. So we see that the map f is not surjective if and only if there is some group extension of C_L by G distinct from the extensions generated by the class of the one given by $\mathrm{Gal}(L^{(1)}/K)$. Hence the theorem is proved. ∎

We close by noting that this theorem says that extra capitulation in a cyclic extension occurs not because of the type of extension of C_L by G given by $\mathrm{Gal}(L^{(1)}/K)$, but because there exists other extensions of C_L by G as well. Further work may show that a correct interpretation of Theorem 5 would imply that Galois realizations of these other extensions are possible, but require l-ramification for each rational prime l dividing n; that is, ramification of those primes which lie over the rational prime l which divide n.

References

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